A Conceptual Approach to Building the Rectangular Area and Rectangular Prism Volume Equations in a Fourth Grade Classroom

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Abstract

The author describes a conceptual, two-day teaching strategy addressing the area of a rectangle and the volume of a rectangular prism. The purpose of the study was to determine if students could generate these formulas. The study was completed in a fourth grade classroom in which students would be engaged in tasks finding these measures, first conceptually with paper square inches and wooden square inches, then with a challenge to relate the physical to the conceptual meaning of multiplication (groups). Once the connection was made, students were then challenged to find area and volumes using the standard algorithms. Eight weeks later, a follow up was completed with students to determine what they remembered. Students, even those with less mathematical confidence and achievement, were able to recall much of the conceptual learning though no follow up tasks were undertaken during the interim.

Keywords: Conceptual Understanding, Area, Volume, Rectangle, Rectangular Prism

1. Introduction

Length, weight, volume – if any area of mathematics lends itself well to hands-on learning, it should be measurement. Yet, I can recall little use of concrete materials when I studied these concepts, even when in college to become a mechanical engineer. And despite all of my successful paper-pencil tasks, I had little *experiential* knowledge of what exactly it was that I was doing. For example, I really didn't know what a cubic yard of something looked like. Or why 12 inches composed a foot, but 144 square inches composed a square foot (why not 12 square inches?). I was just told that this was what was correct.

When I pose these same questions to students in my mathematics methods classes, they are usually at a loss for answers. Though *Principles and Standards for School Mathematics* (2000) states that third through fifth graders "should develop strategies for determining surface area and volume on the basis of concrete materials" (p 175), my students have difficulties conceptualizing any way of solving such problems unless they rely upon a standard algorithm.

So, when I had the opportunity to discuss how I could help Mrs. Burkart teach measurement to her fourth-grade class, I kept in mind the importance of incorporating lessons that were hands-on and experiential. When we met, Mrs. Burkart and I listed the strengths and challenges of her students in measurement, identifying two particular needs to address: conceptually understanding the rectangular area formula ($A = l \text{ x } w$) and the rectangular prism formula ($V = l \text{ x } w \text{ x } h$). To begin our work, we examined what researchers of these topics had learned and recommended.

2. Research

The research agreed with many of my experiences in school. Most students' initial exposure to area is often abstract and procedural – basically only working with the formula (Strutchens, Martin, & Kenney, 2003). Students develop little conceptual understandings such as when using square units to solve problems (Lehrer, 2003). Children from procedurally based classes tend to compare the magnitudes of areas from different shapes by mistakenly comparing corresponding linear measures from each shape (such as comparing the longest sides of various rectangles), believing that the longer measures translate to greater areas (Mullet and Paques, 1991). Similar findings for volume were noted.

And though teachers often thought of finding rectangular area as a “simple task” of multiplying length and width (Lehrer, 2003), researchers noted that finding area and volume was much more complex; students require many years of exposure to varied *hands-on tasks* to develop a complete understanding of the content (Stephans and Clements, 2003).

Thus, focusing strictly on abstract formulas is not the most appropriate method to teach area and volume (Carpenter, et all, 1975). Instead, the literature recommends that students be actively involved in constructing measurement concepts *before* considering the formulas (Kamii and Clark, 1997; Wilson and Rowland, 1993). For example, one study found children employing tiling (covering rectangles with square units) developed strategies that more accurately determined the greater region than students from a traditional classroom; furthermore, these strategies conceptually represented the multiplicative aspect of the area formula (Mullet and Paques, 1991). Another study reported how students working with visual objects developed their spatial proportional reasoning, essential for finding the surface area of three dimensional solids (Obara, 2009). Diagrams alone were found inadequate; students require tasks that use *materials* for finding areas and volumes (Outhred and McPhail, 2000). It was recommended that students reflect upon their conceptual constructions and discuss them as a class, merging the conceptual with the abstract (Wilson and Rowland, 1993). To be clear, the research makes a point of incorporating conceptual work *prior* to abstract formulas; for example, difficulties were reported when attempting to help students construct conceptual understanding of volume once they had been exposed to the volume formula (Tekin-Sitrava and Isiksal-Bostan, 2014).

3. Teaching Strategy

In the research, we found a strategy for teaching rectangular area that was in line with what we had learned (Stephan and Clements, 2003). For our area lesson, we would first allow students to use square units to cover the rectangles, making a point to discuss the necessity of not leaving any gaps or overlapping units when tiling. We then would ask the students to examine these rectangular arrays, encouraging them to find “groups” of tiles coinciding with the length and width of the rectangles. Then we would ask students if they could see a relationship between these groups and their conceptual understandings of multiplication – thereby developing the area formula for rectangles; this direct connection seems to be missing from the literature, as conceptual methods are emphasized but not methods for students to use this work to generate the algorithm. This would be a major objective for our study.

We also found a strategy for “packing volume” (the use of cubic units to fill a rectangular prism). Researchers emphasized the idea of “layers”: filling a prism with cubic units, noting the number of layers that compose the space inside, then using multiplication to find the number within each layer as well as the total number of units (Battista and Clements, 1996). Again, though conceptual work with cubic units is described, how do we get students to generate the formula? This would be a second objective for our study.

We used these strategies to inform our work, which resulted in the following lessons of March, 2015. These lessons would be the first area and volume tasks for our students since the prior academic year. At that time, the students had a very traditional, procedurally-based introduction to these concepts.

4. The Lesson: Day 1

Mrs. Burkart passed out several items which excited the students (figure 1): glue sticks; scissors; a ruler with both inch and centimeter markings; a sandwich bag containing approximately 32 pink, square inches and 10 purple, square centimeters; and the handout titled “Rectangular Shapes.” The handout was written to guide students so they could work independently or in groups.

Students began by examining the square inches and centimeters to determine why they were named as such. Next, students were asked to draw an 8" line, create a rectangle of 8 square inches, and then compare the two shapes – what was similar and different about them? They also were tasked with drawing a line of 6 cm and a rectangle of 6 square cm for the same purpose. The point of these activities was to help students understand how different units of the same value (such as 8 inches and 8 square inches) form completely different shapes and the corresponding importance of labeling answers with the proper units. (In Day 2, this same tactic is used to distinguish cubic units).

Then, students were asked to cover three different size rectangles with their square inches and glue them so that the units remained firm (figure 2). Students were asked if they could find columns and rows – "groups" – and to then find a relationship between the groups and the total number of units. Students successfully saw this relationship, and were able to generate the traditional area formula for a rectangle (though without the precise terms, "length" and "width"). Students were reminded of the formal formula that they had learned a year earlier ($A = l \ x \ w$), and students were able to integrate their past knowledge with this conceptual understanding. The deliberate connection completed one objective of the teaching strategy.

Finding areas of composite shapes are often difficult for students. Typically, students are given composite shapes which are not to scale. The teacher then shows how to divide the shape into smaller rectangles, find the area of each rectangle, then sum the measures to find the area of the composite shape. In our lesson, we used hands-on techniques and a full-scale rectangle to help students conceptually understand this abstract method.

Students took one of their pasted rectangles, cut out part of a column, and moved this part to a new position – forming a composite shape. The teacher asked the students to compare the areas of the original rectangle and the composite shape. “They’re the same!” said several students, “We can see that.” The teacher asked how they arrived at this conclusion. Several students looked a bit exasperated! Several replied with answers such as, “We just moved a piece!” and “Nothing changed!”

The students then drew lines on their composite shape, forming a set of rectangles. Some drew horizontal lines, others drew vertical lines. Then the students counted the units within each rectangle and summed them together. The teacher asked different students to share their work with the class, and asked them to compare the sum with the area of the “original” composite shape. The students seemed to understand that the area of the rectangle and the composite were equivalent, and better understood how “cutting” a composite shape into rectangles could yield the total area. (Our lesson included only one such task since the teacher wished only to introduce the concept. This task, along with similar concrete tasks that then move to the traditional, paper-pencil abstract method - using shapes not to scale - would be very suitable for study in higher grades.) This concluded our work for the first day.

5. The Lesson: Day 2

We began Day 2 with a review of the first day’s work, noting square units, groupings, and the multiplicative method to find the area of a rectangle. Then we moved to a new task – finding the volume of various boxes using cubic inches and cubic centimeters. Again, we assumed nothing; we began by posing the question of how difficult it would be to find the volume of a box using paper square inches. As with the square inch and square centimeter comparison of the day before, students were asked to compare the cubic inches with the cubic centimeters. Then the students were given the definition of a rectangular prism, provided four prisms, and asked to estimate the number of cubic inch units it would take to fill “Rectangular Prism 1.” A brief discussion focusing on the method more than the answers ensued. Then students filled the prism, found the volume, and compared their answer to their estimate. The teacher walked around inquiring why their estimates were different than their answers. Several noted how they only considered one side of the prism or only one layer. After this, students were asked why the units were labeled “cubic inches” and not “square inches”: “We use the blocks – the cubes – not the paper inches,” was one girl’s response. Also, students were asked if the units could be labeled “cubic *centimeters*” and students seemed to be able to differentiate between the units.

Mrs. Burkart then had a little fun with the students; now she wanted to emphasize the importance of writing the correct label for their answers. “Show me four,” she said, and walked around the room. Several students looked up and stared at her. Some muttered, “Four what?” “Show me four!” replied Mrs. Burkart. As she walked around the room, students would show her four cubes; “No, I want you to show me four!” she said. Several students quickly drew a four inch line, some gathered four paper squares, and all motioned for her to come see their work. “Nope!” came the reply. More students began to get frustrated and more openly asked, “Four *what*?!” Mrs. Burkart was clearly enjoying herself as she walked around the room! “Why can’t anyone show me the correct answer?” she asked. The class broke out in a small pandemonium! “We don’t know what you want!” “You need to tell us what you want us to show you!” “Right!” exclaimed Mrs. Burkart smiling broadly. “I need to tell you more than the number, don’t I! So, when you put write an answer, what do you need to do for *me*?” she asked. The students got the point!

Next, students estimated the volume of “Rectangular Prism 2” and then filled it. Then they were asked to remove the cubes “intact” and set them outside the prism. The goal was for students to more easily see “layers” of cubes, each layer being composed of equal groups. Students were challenged to first use multiplication to find the number of cubes in each layer (l x w) and then the total number of cubes that composed the prism (multiplying (l x w) by h). “Look at your cubes,” directed Mrs. Burkart. “Look at the top layer. Can you see groups? Show me.” Mrs. Burkart moved around the room asking students to show her their groupings. “Good! You see the groups. How can you use multiplication to find the total number of cubes in a single layer?” she asked, motioning back and forth with her hand to indicate a horizontal layer. “Multiply” was the general consensus and Mrs. Burkart moved on. “If you have eight cubes in a layer, can you use math to find the total number of cubes in the entire prism?” Some students wished to add the layers and Mrs. Burkart supported their answers. However, she highlighted the answers claiming “times” or “multiply,” asking if this would be a faster method than addition. Finally, Mrs. Burkart asked the students to put it all together: how could they find the number of cubes in their rectangular prism by only using multiplication? One particularly bright boy raised his hand. “You can multiply the top, the top cubes, and get the answer and then take that and times it by the number of layers.”

Though the terms are different ("number of groups", "number in each group", and "layers" vs. "length", "width", and "height"), the conceptual underpinnings of the volume equation seemed apparent from student justifications. Then students continued to explore the concept of volume by examining two more prisms – and using another cubic unit (cubic centimeters in Rectangular Prism 4). As each volume was found, the teacher walked around and asked different students to describe how to use multiplication to find their answers and to show her with the blocks why they thought multiplication applied. With each task, it seemed that more students were able to conceptually describe the volume equation. By understanding the concept of volume through the hands-on tasks, students seemed ready to find volume more abstractly – and they were. Students had little difficulty finding volumes of drawings not drawn to scale; our second objective had been achieved.

It is important to note that throughout this strategy the teacher did not begin with or quickly jump to abstract methods. As in most mathematics, students usually need time to concretely explore the concepts before they are ready to tackle abstract problems. Even for students who can memorize and find the correct answers (like me), often they don't understand how the formula "makes sense" and what exactly it is that they are finding. This general method of teaching, beginning with concrete tasks that naturally lead to abstract methods constructed by the students, is a much more powerful and meaningful way to teach than asking students to memorize and apply formulas. What's more, students will see that math makes sense, is meaningful, and is NOT some type of unreasonable system of formulas and routines to be memorized for some unknown reason.

7. Follow-Up, Further Work, and Conclusions

We were pleased with how the lessons unfolded. However, could the students recall the conceptual understandings of rectangular areas and volumes after a significant amount of time had elapsed? I returned to the classroom about 8 weeks later to understand what students of various academic abilities retained. Mrs. Burkart set up interviews with two of her "higher" achieving students, two "average" students and two students that typically struggle in mathematics (however, I was only able to successfully interview one struggling student; the other seemed very disinterested in my questions and made it clear that he had "no need" for mathematics!). I was a bit anxious given all of the time that had passed, especially since there had been no classroom follow-up in this area. I had several square inches, square centimeters, cubic inch blocks, and cubic centimeter blocks as well as a few drawn rectangles and rectangular prisms available. I asked each student if they could recall the lesson that used these items, and what they could remember about the lesson. The higher and average achieving students did exceptionally well describing the area and volume concepts using the manipulatives provided; they were able to recall the groupings within each concept and connect these groupings to multiplication (the area and volume formulas). Though the struggling student initially did not recall even using the items in class, given a little amount of time she had few problems finding areas and volumes using the manipulatives and connecting the work to multiplication and groups. Overall, I was very impressed with *all* of the students!

The only area of my interviews that I found troubling was in the use of terminology. Though the students had no problem identifying the correct units for each task, and describing why the units were what they were (such as square centimeters or cubic inches), they would use the terms "area" and "perimeter" almost interchangeably, as well as calling a rectangle a "box." When finding volume, one student used cubic inches and described the bottom layer in terms of area and perimeter ("There's 4 x 5 in the perimeter" [meaning the volume of the first level of the prism]).

As students build new concepts, often there are some misconceptions that may arise. Nevertheless, I discussed the terminology issue with Mrs. Burkart. For next year, we decided to add another component at the end of this strategy: using open-ended questions to help students clarify their thinking with these concepts. The strategy is to ask students to think about "area," "volume," and "perimeter" and list on the board characteristics of each. Once completed, Mrs. Burkart will challenge students to compare each type of measurement and find at least three things each have in common, and three things that make each different. After the class discusses various answers, Mrs. Burkart will ask the students to find several examples of each measurement or each figure in their daily lives, such as finding the measure around a picture frame or filling a box full of toys. In this way, it is hoped that students will develop a deeper appreciation for the meaning of each term.

However, overall it seemed the students found the tasks relatively easy and interesting. We were very pleased with the lesson as it unfolded, as well as the learning outcomes displayed by the students both during the lessons and on the test. Employing research for how students think and learn measurement concepts keyed a very successful area and volume unit!

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